J. Fluid Mech. (1969), vol. 38, part 2, pp. 315–333
Printed in Great Britain

The strength of vortex and swirling core flows

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(Received 22 January 1969)

This note presents a discussion of the roles of axial momentum flux, flow force, angular momentum flux and circulation in determining the *strength* and hence characterizing the structure of such narrow rotating axisymmetric core flows as swirling jets, vortex jets, sink vortices and vortex wakes. The salient (though sometimes neglected) features of these core flows are that perturbation pressure plays an essential role both in the coupling of axial and azimuthal velocity fields and in the transmission of force along the core, and that flux of angular momentum is invariant only along cores with zero gross circulation. A number of existing solutions are brought into relationship by the discussion, including Long's similarity solution for draining vortices and Reynolds' dimensional treatment of swirling wakes.

Introduction

Reynolds (1962), in a note on similarity in swirling turbulent core flows, has demonstrated the role of dimensional arguments for deducing asymptotic structures of swirling turbulent wakes. However, several obscurities remain in his paper because of the undue brevity of treatment, including among other factors the assertion that a swirling wake is characterized by its fluxes of axial and angular momentum (supposed to be flow invariants by Reynolds), the general decoupling of axial and azimuthal flow associated with his disregard of the pressure field, the validity of his regime of 'control by angular momentum' in which the ratio of axial to angular momentum flux is locally small, and the whole role of circulation measured in circuits linking the core. Many of these difficulties can be clarified by a discussion of narrow rotating axisymmetric core flows based on integrated forms of the equations and on order-of-magnitude arguments. The results obtained from this discussion must have been known to the classic hydrodynamicists but are not common in the literature and appear to have escaped the attention of a number of recent workers on concentrated vortices, particularly those concerned with atmospheric applications. The following discussion is given primarily for laminar flows, although in most cases the extension to turbulent core flows may be seen without difficulty.

The strength of a *jet* produced in an extensive uniform environment otherwise at rest is normally taken as the flux of axial momentum from its source, as this

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is the only quantity invariant under transport by the jet; and since dynamic pressure variations are small in laminar jets (and balanced in turbulent jets) apart from some neighbourhood of the source, it is also the effective force of the jet. The *wake* formed behind a compact body moving in an otherwise still environment is also characterized by axial momentum flux equal and opposite to the drag force on the body, or in the case of a fixed body in a uniform stream by flux of momentum deficit equal to the drag.

The principal flow function for both jets and wakes might usefully be regarded as the transmission of force through the fluid. Indeed, perhaps the best definition for a wake is that of the disturbance flow serving to transmit drag–force reaction from a moving body through the surrounding fluid and ultimately to the boundaries. This definition encompasses both the usual 'viscous wakes' behind bluff bodies and also 'wave wakes' in cases where a body experiences wave-making resistance, as on a free liquid surface or in a stratified rotating fluid; wave wakes will not be considered further here. It is instructive to regard the transmission of flow force as the prime function of a variety of core flows generated effectively from a source, sink or doublet, where by *core flow* we imply a narrow flow subtending a small angle of spread from the neighbourhood of its virtual origin. In discussing the characterization of such flows as swirling jets, vortex jets, sink vortices and vortex wakes we shall in each case define the *flow strength* as the conserved force quantity or quantities or other invariant of the flow. We may note that while force is often conserved in narrow flows in otherwise undisturbed environments, mass conservation typifies the normal sink flow and the idealized (though almost unrealizable in practice) symmetrical source flow.

As a basis for discussion we require integrals of the flow equations and orders of magnitude of the dependent variables for regions of settled flow excluding the immediate neighbourhood of the source, and we shall deal first with laminar core flows produced from fixed sources in extensive environments at rest at infinity.

Laminar cores in a still environment

The equations for steady, incompressible, axisymmetric, laminar flows referred to cylindrical polar co-ordinates (r, θ, z) with origin fixed in position relative to the source, and with velocity components (u, v, w) tending to zero in the relatively undisturbed fluid far from the source, take the forms

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0, \tag{1}$$

$$\frac{U}{R} \sim \frac{W}{Z},$$

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}\, ru + \frac{\partial^2 u}{\partial z^2}\right), \tag{2}$$

$$\frac{U^2}{R} \sim \frac{UW}{Z} \quad \frac{V^2}{R} \qquad \frac{P}{\rho R} \qquad \frac{\nu U}{R^2} \gg \frac{\nu U}{Z^2}$$

$$u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=\nu\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}rv+\frac{\partial^2 v}{\partial z^2}\right), \tag{3}$$

$$\frac{UV}{R}\sim\frac{VW}{Z}\sim\frac{UV}{R}\qquad\frac{\nu V}{R^2}\gg\frac{\nu V}{Z^2}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right), \tag{4}$$

$$\frac{UW}{R}\sim\frac{W^2}{Z}\qquad\frac{P}{\rho Z}\qquad\frac{\nu W}{R^2}\gg\frac{\nu W}{Z^2}$$

where capital letters represent characteristic magnitudes for the variables in a slice of the core distant approximately Z from the virtual source and of 'core radius' R, suitably defined, and appropriate orders of magnitude have been entered under the terms of the equations. Boundary conditions on the flow in a plane of section of the core are:

$$\left.\begin{array}{l}\text{at}\quad r=0,\quad u=0,\quad v=0,\quad w\,\text{finite},\quad \partial w/\partial r=0,\quad p\,\text{finite},\quad \partial p/\partial r=0;\\ \text{as}\quad r\to\infty,\quad u\to 0,\quad v\to 0,\quad w\to 0,\quad p\to 0;\end{array}\right\} \tag{5}$$

where p is the disturbance pressure due to the core flow. We note that it is not possible at this stage to impose general requirements on the velocity moment ru as $r\to\infty$.

Commonly observed vortex and swirling core flows are narrow, with semi-angle of spread $R/Z=\alpha \doteqdot 0{\cdot}1 \ll 1$, and the axial viscous diffusion is in such cases negligible relative to lateral diffusion. From (1)

$$U\sim\frac{RW}{Z}\sim\alpha W; \tag{6}$$

and from (3) either the dominant viscous and inertial forces are in balance with

$$\alpha\,Re=\frac{R}{Z}\frac{RW}{\nu}\sim 1, \tag{7}$$

in which case the flow is likely to be laminar, or the Reynolds number

$$Re=RW/\nu\gg\alpha^{-1}\gg 1$$

is large and the flow will be turbulent, and for the present outside our attention. From (4) for laminar flows, either (i) P/ρ enters directly into the inertial–viscous balance with

$$P\sim\rho W^2, \tag{8a}$$

or (ii) the axial pressure gradient plays a minor role in the equation and

$$P\ll\rho W^2. \tag{8b}$$

Finally, from (2), the ratio of inertial to centrifugal to pressure gradient to viscous forces is

$$\alpha^2:\frac{V^2}{W^2}:\frac{P}{\rho W^2}:\alpha^2;$$

hence in case (i) with $P\sim\rho W^2$ equation (2) reduces to a centrifugal–pressure gradient force balance with

$$V\sim W. \tag{9a}$$

For $V/W \sim 1$ the pressure perturbation is due directly to the azimuthal flow, but provides strong coupling with the axial flow. As V/W decreases, the dynamical role of the pressure declines progressively until $V \sim U$ or smaller, after which $P \sim \rho U^2$ and is independent of V, with complete decoupling of the axial and azimuthal velocity fields; such weakly rotating cores merely provide forced convection of angular momentum without significant modification of the basic axial–radial flow. Slightly different order-of-magnitude arguments apply for swirling and vortex wakes, but these do not affect the conclusion that $V \sim W$ as a result of close pressure coupling for strong rotation.

Rearrangement of relations (6)–(9) yields the scales: (i) for strong rotation

$$U \sim \frac{\nu}{R}, \quad V \sim W \sim \frac{\nu Z}{R^2}, \quad P \sim \rho\left(\frac{\nu Z}{R^2}\right)^2;$$

and (ii) for the weakly rotating case

$$U \sim \frac{\nu}{R}, \quad W \sim \frac{\nu Z}{R^2}, \quad P \sim \rho\left(\frac{\nu}{R}\right)^2,$$

with V to be determined from the rotation structure imposed on the core. In order fully to determine the structure of the core flow in terms of the single independent variable Z one further relationship is needed, and this can be obtained from a strength relation for the flow representing the dominant effect of the source.

The gross transport properties of narrow core flows may be derived without further approximation by integrating the equations of motion over a plane section z = constant. From (1),

$$\frac{d}{dz}\int_0^\infty \rho r w\, dr = -(\rho r u)_\infty, \tag{10}$$

where $(\rho r u)_\infty = \text{limit}_{r\to\infty}(\rho r u)$; thus the axial rate of increase in mass flux is equal to the rate of mass entrainment or radial inflow from large distances.

Integration of (2) provides a relationship for the pressure defect on the axis,

$$[p]_0^\infty = \rho\int_0^\infty \frac{v^2-u^2}{r}\,dr - \frac{d}{dz}\left(\int_0^\infty \rho u w\, dr + \rho\nu w_0 - \rho\nu\int_0^\infty \frac{\partial u}{\partial z}\,dr\right), \tag{11}$$

where $w_0(z) = w\ (r=0, z)$. Using the scales derived above, this reduces for the case of strong rotation to

$$p_\infty - p_0(z) = \rho\int_0^\infty \frac{v^2}{r}\,dr,$$

with relative error of order α^2, corresponding to a radial pressure gradient–centrifugal acceleration balance; while for weak rotation most terms survive, but pressure variations as a whole are small ($P \sim \rho\alpha^2 W^2$) and may be neglected. We note that the pressure $p_\infty - p_0$ on the axis of a rotating core embedded in an externally still environment at uniform pressure p_∞ is constant only in a cylindrical field $v = v(r)$ of azimuthal velocity, and that there will always be axial-pressure gradients (with corresponding axial velocities) in rotating cores exhibiting axial development of the azimuthal velocity field $v = v(r, z)$. Thus *axial pressure gradients are the rule in all swirling flows suffering progressive lateral spread by either viscous or turbulent diffusion* with increasing axial distance from the source.

Equation (3) yields the azimuthal momentum integral,

$$\frac{d}{dz}\int_0^\infty \rho r\left(vw - \nu\frac{\partial v}{\partial z}\right) r\,dr = -(\rho r^2 uv)_\infty + \left(\rho\nu r^3\frac{\partial}{\partial r}\frac{v}{r}\right)_\infty, \tag{12}$$

showing that the axial rate of change in the flux of angular momentum (due primarily to convection, and in negligible proportion α/Re to axial diffusion) is the result of an effective torque due to entrainment of azimuthal momentum at infinity and to the moment of the viscous stress at infinity.

Finally, from equation (4),

$$\frac{d}{dz}\int_0^\infty \left(\rho w^2 + p - \rho\nu\frac{\partial w}{\partial z}\right) r\,dr = -(\rho ruw)_\infty + \left(\rho\nu r\frac{\partial w}{\partial r}\right)_\infty. \tag{13}$$

Contributions to the force transmitted by the flow from the direct flux of axial momentum, the disturbance pressure and the axial diffusion of momentum are in the ratio $1:(P/\rho W^2):\alpha^2$; hence the contribution from viscous diffusion is always negligible and that from pressure thrust is important in strongly rotating but negligible in weakly rotating cores. In strongly rotating cores the pressure thrust is of comparable importance to the momentum flux, and the axial pressure gradients normally formed result in a progressive transfer of force between the momentum flux and pressure fields. In such cases the *flow force*† per unit section,

$$f = \rho w^2 + p,$$

plays a more fundamental role than the simple momentum flux. Equation (13) now has the direct interpretation that in the absence of body forces the axial changes in flow force are due to entrainment of ambient axial momentum and to outer viscous stress.

Certain general remarks can now be based on (10)–(13) provided that estimates can be obtained for $(ru)_\infty$ and $(rv)_\infty$. All viscous (or turbulent) axial core flows exhibit entrainment in the sense that neighbouring ambient fluid gains axial momentum by diffusion, is accelerated into the core flow, and (in a uniform still environment) is replaced by more distant ambient fluid driven inwards by the weak external pressure field generated by acceleration into the core. Thus the asymptotic effect of the core flow at large radial distances is effectively that of some distribution of sinks along the axis of symmetry, and the asymptotic radial inflow velocity in an otherwise still, uniform environment must be $O(r^{-1})$, and $(ru)_\infty$ is finite and generally a function of z. Taylor (1958) has previously utilized

† There seems to be little uniformity in the use and naming of f, though it is important in swirling flows, rotating systems and stratified flows (where an analogous flow force involves the perturbation pressure associated with density perturbations). The term 'momentum flux' is sometimes used for ρw^2 and sometimes for $\rho w^2 + p$, but there is a case for calling $\rho w^2 + p$ the *flow force* and ρw^2 the momentum flux. The term 'flow force', which enters naturally here, has been used recently by Benjamin (1962).

this approach in calculating some entrainment flows induced around jets and plumes.

The asymptotic value $2\pi(rv)_\infty$ is precisely the circulation K_∞, and is a measure of the net axial vorticity of the core flow. At this stage we might note that the flows under consideration have small angles of spread and are of boundary-layer (core) type. Thus r has more or less the character of an inner variable, and $r \to \infty$ does not imply $r/z \to \infty$, but merely that r is sufficiently large to include all of the core region. While K_∞ must be independent of z in a still environment, it may, over limited regions, vary with axial distance in the case of a streaming outer flow, as in the rolling up of a trailing vortex sheet above a swept wing to form a trailing vortex core with progressively increasing circulation (Hall 1966). It is essential to distinguish between the flux of angular momentum in the core and the circulation around the core. A swirling jet generated by release from an orifice of fluid with both axial and angular momentum will have circulation $K(r)$ in circular paths steadily increasing with radius r to a maximum $K_m(r_m)$, and then decreasing to zero for some $r_a(z) > r_m(z)$; the vorticity will be of opposite sign in $0 < r < r_m$ and $r_m < r < r_a$.

It will be convenient to divide rotating core flows into *swirling* flows with $K_\infty = 0$ and *vortex* flows with $K_\infty \neq 0$. We shall consider some particular cases.

The laminar jet

$-2\pi(ru)_\infty = V_e(z)$, $(rv)_\infty = 0$, $w_\infty = 0$, where $V_e(z) > 0$ corresponds externally to the volume flow per unit axial length into an axial distribution of sinks with strength, $q(z)$, decreasing slowly with increasing z. The mass flux along the core,

$$\int_0^z q(\theta)\, d\theta,$$

increases monotonically with z. The pressure contribution to the flow force may be neglected as $P \sim \rho U^2 \sim \alpha^2 W^2 \ll W^2$, and it follows from (13) that the momentum flux is constant,

$$2\pi \int_0^\infty \rho w^2 r\, dr = F, \qquad (14)$$

where F is to be taken as the strength of the jet. From (14), $RW \sim (F/\rho)^{\frac{1}{2}}$, and the jet is characterized by the local (and over-all) Reynolds number

$$RW/\nu \sim (F/\rho\nu^2)^{\frac{1}{2}};$$

the asymptotic structure of the flow is now determined as

$$R \sim (F/\rho\nu^2)^{-\frac{1}{2}} Z, \quad U \sim (F/\rho\nu^2)^{\frac{1}{2}} \nu Z^{-1}, \quad W \sim (F/\rho\nu^2)\, \nu Z^{-1}, \quad P \sim F Z^{-2},$$

and the Reynolds number $(F/\rho\nu^2)^{\frac{1}{2}}$ must be large for a narrow core $\alpha \ll 1$, so that the jet will normally be turbulent. Solutions for the laminar jet are well known, and were given originally by Schlichting (1933) in boundary-layer approximation, and by Squire (1951, 1952) without approximation.

The viscous line vortex

$u \equiv 0,\ 2\pi(rv)_\infty = K_\infty,\ w \equiv 0$, is a two-dimensional flow with vorticity components

$$\left(0, 0, \frac{K_\infty}{4\pi\nu t} e^{-r^2/4\nu t}\right),$$

and velocity components

$$\left\{0, \frac{K_\infty}{2\pi r}(1 - e^{-r^2/4\nu t}), 0\right\},$$

arising from the (hypothetical) release at time $t = 0$ of a concentrated line vortex on the z axis. This is a time- rather than space-dependent flow, but must be included here as the archetype of all core flows with circulation. The solution (Lamb 1932, p. 591) is often given in terms of vorticity directly from the analogous problem of heat conduction from an instantaneous line source, but this must not be allowed to obscure the fact that the changing velocity distribution associated with spreading of the vorticity field is due to a *stress* distribution. It is readily shown that the fluid contained in a disk of unit thickness normal to OZ and of radius $r = R$ has:

$$\left.\begin{array}{ll} \text{circulation} & K_\infty(1 - e^{-R^2/4\nu t}), \\ \text{kinetic energy} & \dfrac{1}{8\pi}\rho K_\infty^2\{\log R + \gamma - \log 2 + O(e^{-R}/R)\}, \\ \text{angular momentum} & \tfrac{1}{2}\rho K_\infty\{R^2 - 4\nu t(1 - e^{-R^2/4\nu t})\}. \end{array}\right\} \quad (15)$$

The kinetic energy of an infinite disk of fluid is logarithmically infinite; and its angular momentum is algebraically infinite but decreases with time at finite constant rate, from which it follows that a steady retarding torque or couple acts on the fluid at infinity. The moment of the tangential stress acting at distance r is

$$2\pi r\sigma_{r\theta} r = 2\pi\rho\nu r^3 \frac{\partial}{\partial r}\left(\frac{v}{r}\right) = -2\rho\nu K_\infty\left\{1 - \left(1 + \frac{r^2}{4\nu t}\right)e^{-r^2/4\nu t}\right\};$$

thus the retarding torque at infinity is transmitted inwards by a continuous distribution of retarding tangential stress which produces the progressive retardation and spreading out of the central azimuthal velocity field.

Single concentrated line vortices cannot be generated because of their infinite energy and angular momentum; however, a pair of distinct line vortices with equal and opposite strengths $\pm K$ and separation $2d$ has finite kinetic energy, zero net angular momentum and zero net circulation. Such vortex pairs, often with one member an image vortex, can be generated readily but in normal circumstances are found to decay rapidly by the growth of axial flow. Termination of the individual vortices at a rigid boundary is impossible as the balanced centrifugal pressure field of the vortex is disrupted in the end-wall boundary layer, so that ambient fluid is driven into the low-pressure vortex core which fills rapidly along its whole length by axial inflow. A tolerable estimate for the decay time of a straight line vortex produced between the side walls of a tank of water can be obtained using the axial velocity scale $W \sim V$. The sole effective termination is at a density-discontinuity surface in a multi-layered liquid with immiscible components, in which case it may be possible to produce vertical sections of a vortex or of vortex pairs approximating in local behaviour to the viscous line vortex.

The vortex pairs observed most commonly are vortex wakes or other cores with significant axial motion, but these preserve the principal azimuthal flow structure features of the viscous line vortex within half the separation distance ($2d$) of the cores provided that the diffusion length $\surd(4\nu t) \ll d$. There will be slow loss of circulation from each core by diffusion of vorticity of opposing sign across the plane of symmetry separating the two cores, though this effect will be relatively small provided that the individual cores are well separated. However, the retarding tangential stress of a viscous line vortex will always be important in cases in which the loss of circulation is small; hence the flux of angular momentum is not a conserved quantity in individual vortex cores with axial flow.

Swirling laminar jets

$-2\pi(ru)_\infty = V_e(z)$, $(rv)_\infty = 0$, $w_\infty = 0$, where V_e is again the, as yet undetermined, volume entrainment per unit length of axis. The axial mass flux is monotone increasing (equation (10)), and from (12) and (13),

$$\left.\begin{aligned} 2\pi\rho\int_0^\infty r^2 vw\,dr &= G, \\ 2\pi\int_0^\infty (\rho w^2+p)\,r\,dr &= F,\end{aligned}\right\} \tag{16}$$

where both the flux of angular momentum G and the flow force F are conserved quantities (or flow invariants); longitudinal viscous diffusion terms have been neglected on the assumption α small. The order-of-magnitude analysis has shown already, equation (8), that p cannot exceed ρw^2 appreciably: hence from relations (16),

$$R^3VW \sim G/\rho \quad \text{and} \quad R^2W^2 \sim F/\rho, \tag{17}$$

and

$$\frac{V}{W} \sim \frac{G}{RF}. \tag{18}$$

Both G and F are conserved in a swirling jet, and it follows that the local dimensionless parameter G/RF decreases monotonically with increasing downstream distance from its source value G/R_sF; this parameter, appropriately termed the *swirl number*† of the flow, is a ratio of azimuthal to axial velocity scales at entry, and characterizes the effects of axial rotation in swirling core flows.† In strongly swirling flows, from (11), $P \sim \rho V^2$ and

$$\frac{P}{\rho W^2} \sim \frac{V^2}{W^2} \sim \left(\frac{G}{RF}\right)^2.$$

† Two different parameters have been used to characterize the effects of rotation on flows. (i) The Rossby number is the ratio of relative inertial to Coriolis forces in a fluid body undergoing rotation as a whole; or, equivalently, the ratio of relative velocity in a plane normal to the appropriate axis of rotation to a typical transverse velocity of back-

The azimuthal velocity cannot exceed the axial velocity in order of magnitude because of the close pressure coupling: hence the swirl number cannot exceed order unity in unconfined core flow, except possibly in some neighbourhood of a free axial stagnation point, and the disturbance pressure and axial momentum flux make comparable contributions to the flow force. Quite large adverse gradients of axial pressure are observed in strongly swirling jets, caused by progressive lateral diffusion of angular momentum with consequent reduction in magnitude of the perturbation pressure deficiency at the axis, and at swirl parameters a little larger than unity a free stagnation point appears on the axis of a swirling jet followed by a downstream bubble of recirculating flow (Gore & Ranz 1964); such cores will normally be turbulent. 'Breakdown' of a swirling jet, associated with stagnation and reversed flow near the axis, causes a sudden increase in jet width and hence a sudden reduction in the local swirl number, and may be regarded broadly as confirmation of the deduction that a free swirling jet cannot support azimuthal velocities much in excess of its axial velocity.

Typical scales for swirling jets are obtained from (6), (7) and (17) as

$$R \sim \left(\frac{F}{\rho\nu^2}\right)^{-\frac{1}{2}} Z, \quad U \sim \left(\frac{F}{\rho\nu^2}\right)^{\frac{1}{2}} \frac{\nu}{Z}, \quad V \sim \frac{G}{\rho\nu}\left(\frac{F}{\rho\nu^2}\right)^{\frac{1}{2}} \frac{1}{Z^2}, \quad W \sim \frac{F}{\rho\nu^2}\frac{\nu}{Z}, \tag{19}$$

with total Reynolds number $(F/\rho\nu^2)^{\frac{1}{2}}$. In strongly swirling jets $P \sim \rho V^2$ plays an essential dynamical role; but in weakly swirling jets $\rho V^2 < P \sim \rho U^2$, and from (18),

$$\frac{R}{Z} \sim \frac{U}{W} > \frac{V}{W} \sim \frac{G}{RF},$$

or

$$\frac{G}{RF}\left(\frac{F}{\rho\nu^2}\right)^{\frac{1}{2}} < 1.$$

Thus the dynamical role of swirl in a jet always suffers progressive decrease in importance with increasing axial distance, and will disappear by the neighbourhood

$$z \sim R(F/\rho\nu^2)^{\frac{1}{2}} \sim G/\rho\nu^2;$$

beyond this the flow degenerates to forced convection of angular momentum in a weakly swirling (but otherwise unmodified) jet. Provided that

$$R_s > k(G/\rho\nu^2)(F/\rho\nu^2)^{-\frac{1}{2}},$$

where R_s corresponds with the source radius and k is a constant of order unity, a jet will exhibit weak swirl throughout; this case has been treated by Görtler

ground rotation in this plane. The Rossby number provides a measure of the constraint on an interior flow due to the rotation of its whole environment; thus balanced geostrophic flow with velocity normal to the pressure gradient is a typical small Rossby number flow, and these flows have a general tendency to be two-dimensional. (ii) The swirl parameter, usually measured as the local ratio of typical transverse and axial velocity components in a swirling jet or wake in a rotation free environment, provides a measure of the disruptive effect of axial swirl, including the tendency to increased rates of spread and the development of flow reversal near the axis of the core. Some authors have identified the swirl parameter as an inverse Rossby number, but this has disadvantages as the original Rossby number characterizes the effect of external rotation on a flow, while the (inverse of the) swirl parameter characterizes the effect of internal rotation.

(1954) as a problem of forced convection reducing to the solution of a single equation for the azimuthal flow. Loitsianskii (1953) has treated the more interesting case in which R_s falls short of $k(G/\rho\nu^2)(F/\rho\nu^2)^{-\frac{1}{2}}$ sufficiently to produce a region of pressure coupled flow; the solution is developed as a power series expansion in z^{-1} (of which the coefficients are profile functions of r/z) up to first-order effects of swirl, but the method applies only to small modification of the axial flow by swirl as the permissible increase in G/R_sF is limited strictly by the requirements of convergence. A treatment by Lee (1965) for the turbulent swirling jet extends rather approximately to somewhat higher values of G/R_sF. No satisfactory theory exists for the most interesting case of $G/R_sF \sim 1$, but experimental studies on the structure of turbulent swirling jets have been reported by Rose (1962) and on stagnation and core recirculation by Gore & Ranz (1964), and Chigier & Beér (1964).

If fluid in rigid body rotation is emitted from a pipe spinning about its axis, the emerging core is sheathed in a vortex sheet which exactly counterbalances the core circulation and which develops to form an annular mixing layer in which there is decreasing circulation with increasing radius and hence likely to be turbulence. We do not expect to generate a vortex jet in this way, of course, because of energy and angular momentum limitations in the outer flow, and it seems as though there is no practical way of generating even a vortex jet pair in a still and irrotational environment.

Draining or sink vortices

If initially irrotational fluid were drained slowly through an orifice from a vessel the flow would remain irrotational everywhere except for fluid which has passed into a viscous layer near a wall. Vortex formation would then be rare and would arise only through the convection of a separated boundary layer towards the outlet, and the draining flow would normally approximate to that of an ideal fluid. In fact, however, all fluids possess the rotation of the earth, and in addition most have filling or other vorticity stronger by an order of magnitude or more. In such circumstances a rapid outflow usually produces no observable amplification of vorticity, but a slow outflow generates a well-defined draining vortex (Andrade 1963). The Rossby number, which may be interpreted here as the ratio of the appropriate component of disturbance vorticity to the background vorticity due to rotation of the fluid body as a whole, provides a measure of the capacity for vortex formation, and in general the smaller the Rossby number (or the slower the outflow) the more vigorous the draining vortex that will form in a given time. Thus the formation of a draining vortex depends expressly on the pre-existence of a vorticity distribution, and vortex formation is unlikely in a really still environment.

The flux of angular momentum is no longer a flow invariant when there is net circulation $K_\infty = K(\infty, z)$ about the core, and the scaling in a sink vortex will be different from that of a swirling jet. Using Stokes's theorem, the difference in circulation measured in circular paths (r, z_1) and (r, z_2) is

$$K(r,z_1) - K(r,z_2) = \int_l \operatorname{curl} \mathbf{v} \,.\, \mathbf{n}\, dS = \int \boldsymbol{\omega} \,.\, \mathbf{n}\, dS,$$

where the surface integral is evaluated over the section of circular cylinder with outward normal $\mathbf{n}$ terminated by the two circulation paths (at z_1 and z_2). The change in circulation with distance corresponds to the lateral divergence of vortex tubes, and taking r large we see that the total circulation $K(\infty, z)$ of a vortex core in an irrotational environment is independent of axial distance z and constitutes the second flow invariant for the system. Thus, subject to the earlier assumption of a dynamically significant pressure field, the R, U and W scales are the same as those for the swirling jet, relations (19), but the invariant circulation $K_\infty \sim RV$, and hence

$$V \sim \left(\frac{F}{\rho\nu^2}\right)^{\frac{1}{2}} \frac{K_\infty}{Z} \quad \text{and} \quad P \sim \frac{F}{\rho\nu^2}\rho\frac{K_\infty^2}{Z^2}. \tag{20}$$

Strong vortices have $P \sim \rho W^2$ or equivalently $V \sim W$, and hence

$$\left(\frac{F}{\rho\nu^2}\right)^{\frac{1}{2}} \sim \frac{K_\infty}{\nu}; \tag{21}$$

and for narrow vortices

$$\left(\frac{F}{\rho\nu^2}\right)^{\frac{1}{2}} \sim \frac{Z}{R} \gg 1. \tag{22}$$

Narrow vortex cores in which the circulation is an increasing function of radius exert a stabilizing influence on their axial core flow and remain laminar to rather higher Reynolds numbers, $(F/\rho\nu^2)^{\frac{1}{2}}$, than comparable jets (in accord with the strong laminar vortices commonly observed). A given vessel of fluid in a specified state of rotation (K_∞)† should exhibit some or all of the following modes of draining, though it must be noted that such draining flow is at most quasi-steady. (i) $(F/\rho\nu^2)^{\frac{1}{2}} \gg K_\infty/\nu, 1$: the ambient vorticity is insufficient to constrain the rapid outflow into a narrow vortex and the flow approximates to three-dimensional sink flow. (Note that the condition $(F/\rho\nu^2)^{\frac{1}{2}} \gg 1$ is not of itself sufficient to ensure a narrow draining flow.) (ii) $(F/\rho\nu^2)^{\frac{1}{2}} \sim K_\infty/\nu \gg 1$: slower outflow will generate a strong narrow draining vortex, and the necessary flow force is determined by the available vorticity. (iii) $K_\infty/\nu \gg (F/\rho\nu^2)^{\frac{1}{2}} \gg 1$: although there is ample vorticity, the rate of outflow is insufficient to maintain a pressure-coupled concentrated vortex. (Reduction of the rate of extraction below an established draining vortex will often destroy the concentrated vortex.) (iv) $(F/\rho\nu^2)^{\frac{1}{2}} \ll 1$: very slow outflow into a weak sink will produce only a modified sink flow.

This discussion serves to emphasize a feature of vortex flow that is perhaps surprising to the bath-time observer, but well known to those who have pursued more careful experiments: that the generation and maintenance of stable vortices may require quite careful selection of the operating conditions‡ (e.g. Turner 1966).

† We recall that K_∞ must be measured in a path lying outside the core but not at $r/z \to \infty$; even though a stationary vessel of fluid stirred into rotation has zero circulation K_w in circuits taken round its wall, it will have non-zero circulation K_∞ in suitably chosen interior circuits.

‡ In many circumstances the draining flow force may adjust as a consequence of modifications in the flow itself to a value appropriate to the existing circulation; the formation of an air core may possibly be such an adjustment.

Vortex jets and plumes can be generated in a rotating environment (Herbert 1965; Morton 1963) under rather limiting conditions, but are experimentally more difficult to handle than draining vortices. A series of studies by Long (1956, 1958, 1961) is amongst the more detailed of a rather variable range of theoretical and experimental investigations of draining vortices and provides some broad confirmation of the foregoing classification of flow regimes in terms of the non-dimensional parameter $(K_\infty/\nu)(F/\rho\nu^2)^{-\frac{1}{2}} = K_\infty(F/\rho)^{-\frac{1}{2}}$, which might be termed the *circulation number*. These experiments were carried out in cylinders of water brought initially to a state of slow rigid body rotation and then drained through an outlet on the axis of rotation at an end plate. Long characterized his rate of draining by a so-called 'Rossby number' (actually the ratio of volume efflux to product of tank radius by circulation measured at the tank wall, which is not a Rossby number in the sense defined above, and in particular cases may be smaller even by orders of magnitude; thus doubling the tank radius without other change for a concentrated draining vortex reduces Long's parameter by almost one order of magnitude for fixed rate of efflux). Long (1956) observed that strong sinks at 'Rossby numbers' exceeding about 0·3 produced three-dimensional draining flows in which fluid was withdrawn from all parts of the cylinder, but that somewhat weaker sinks extracted fluid from a central core only, the diameter of which decreased progressively until intense narrow draining vortices were formed over weak sinks below values of about 0·02 (Long 1958); no observations were reported below 0·006.

Long (1958, 1961) noted from his experiments that once a quasi-steady concentrated draining vortex had developed from the initial slow rigid-body rotation of a tank of water, the vorticity in the core exceeded that in the remainder of the tank by orders of magnitude, so that circulation measured in axially symmetric circuits outside the core changed relatively little with increasing radius. Although the ambient fluid still possesses axial vorticity, this is dominated by the high level of core vorticity; thus the draining vortex should be regarded as a *high* Rossby number flow in which the core has derived its vorticity originally from the ambient rotation but is no longer significantly affected by it in the quasi-steady state. On this basis Long developed a similarity solution for concentrated viscous vortices valid supposedly for both source and sink flows having non-zero circulation K_∞ in an otherwise irrotational environment. However, we have noted already that it is practically impossible to produce a jet with net circulation in an otherwise still environment, and hence Long's solution can have physical validity only for draining vortices in an irrotational or weakly rotational environment.

Long's similarity solution contains several possibly paradoxical features that may merit further discussion. Although the solution is asymptotic in character, it takes the form of a one-parameter family with infinitely many velocity profile shapes, characterized by the circulation number† $K_\infty(F/\rho)^{-\frac{1}{2}}$ which has already been shown to vary over a range from zero up to $O(1)$ but is never large. This is perhaps surprising, as asymptotic regimes of flow normally correspond regions

† Long's parameter M corresponds to the inverse square of our circulation number, and he finds solutions only for values $M > 3{\cdot}65$.

of particular (and usually simple) force balance, whereas the circulation number $(K_\infty/\nu)(F/\rho\nu^2)^{-\frac{1}{2}}$ provides a measure of the dynamical role of the centrifugal pressure perturbation in the flow and the entire force balance of the flow changes as the circulation number is increased from zero to $O(1)$. The family of similarity solutions includes such manifestly different flows apparently because the simple jet (physically realizable), the vortex jet (unrealizable) and the strong draining vortex (realizable) all happen to have conical shape $R \propto Z$, with small angle of spread R/Z provided that the Reynolds number $(F/\rho\nu^2)^{\frac{1}{2}}$ is sufficiently large. However, unconfined sink flows are not narrow in the absence of rotation, and it appears that weak draining vortices at small circulation numbers (though not necessarily small K_∞/ν) must lie outside Long's family of similar flows. The mass flux in each of the remaining similarity flows,

$$M = 2\pi\rho \int_0^\infty wr\,dr \sim 2\pi\rho\nu Z,$$

is monotone increasing in magnitude with distance and zero at the virtual source. These are directed flows, in which the virtual origin is effectively a source of flow force and not of mass, though it seems reasonable to classify them as source vortices if there is net outflow ($M > 0$) and sink vortices for net inflow ($M < 0$) at corresponding actual sources of finite radius. Source vortices may have an inner core of weak reversed flow with slow loss of fluid to the enveloping outflow (and it may be noted that the zero for the radial component tends to lie outside the zero for the axial component of velocity in the solutions with strongest backflow presented by Long (1961)); the increase with distance z of total mass flux in a source vortex may be associated with entrainment of ambient fluid. Reference to Long's solution profiles shows clearly that all of his numerical solutions represent source vortices or vortex jets, and that the net forward mass flux is appreciable even in the case of strongest axial back flow. Thus the constraints imposed by the similarity approach seem to be incompatible with draining vortex flow and to be satisfied only by vortex jets (which are of limited physical significance).

It is possibly worth noting that the contribution to flow force from the centrifugal pressure perturbation is everywhere *negative*, while that from the momentum flux is positive for both forward and reversed axial flow (since the forward transport of forwards momentum and the reverse transport of reversed momentum both correspond to a forward force). Thus for each value of F we should expect (in principle, if not in practice) both a source and a sink vortex; Long also found pairs of solutions for each F, though 'surprisingly' both appear to be source vortices in the cases computed. Thus the pressure and momentum flux contributions to the flow force are opposed, and $P/\rho W^2 \ngtr O(1)$. Unfortunately the methods employed here are not powerful enough to determine whether F must necessarily be positive, though this proves to be so in the cases computed by Long.

The draining vortex model proposed by Long represents a steady vortex in an otherwise irrotational environment from which vorticity is being extracted continuously at the sink. In practice such vortices normally develop in weakly vortical environments, and concentrated cores with vorticity orders of magnitude stronger than ambient seem to develop in the available draining times only when the end of the vortex distant from the sink is anchored at a rigid boundary or free surface. In such circumstances the vortex remains narrow along its length and interacts strongly at the terminating boundary. Such a sink vortex is easily maintained under the balanced effects of inward radial convection (coupled with longitudinal stretching) and outward radial diffusion of vorticity in spite of the vorticity transport to the sink, as in the Burgers (1948) vortex; but there seems to be no comparable mechanism for maintaining source vortices. Although the Burgers vortex (consisting of a concentrated core of vorticity in an irrotational reversed stagnation point flow; see also Sullivan 1959 and Lewellen 1962) is a solution of the full Navier–Stokes equations, it satisfies only *inviscid* boundary conditions at the base boundary and so falls far short of providing an adequate model for vortex termination† at a boundary. In fact, at a rigid boundary, the balanced centrifugal pressure field of the vortex core is disrupted in a thin terminating boundary layer in which a large radial inflow is driven by the unbalanced pressure field. In a tank of moderate depth a considerable proportion of the fluid to be extracted at the sink is likely to enter the core through the distant terminating boundary layer, which will consequently exert a very strong influence on the structure of the entire vortex. Indeed, the diameter of the vortex at termination and hence the magnitude of its centrifugal pressure reduction depend very largely on the terminating flow, and may to some extent be modified by varying the end-wall roughness or configuration. Thus Long's experimental draining vortex may be expected to bear a strong imprint of the flow in the end-wall terminating boundary layer, and is likely to show only a limited relationship to his similarity solution.

Vortex and swirling wakes

The order-of-magnitude analysis for wakes and jets in a uniform main stream of steady velocity W_0 in the direction Oz proceeds like that for a still environment, except that there are two distinct scales for the axial component of velocity w; as a convective velocity $w \sim W_0$, and under differentiation $w \sim W$, the appropriate disturbance velocity scale. For narrow cores we require $R/Z \ll 1$, and sufficiently far downstream of the wake-producing body or jet orifice the disturbance velocity is relatively small, or $W/W_0 \ll 1$. It follows, as before, that $U \sim RW/Z$ and for laminar viscous flow

$$\frac{RW_0}{\nu}\frac{R}{Z} \sim 1, \tag{23}$$

where the naturally occurring Reynolds number in this case is based on the main stream velocity and terms of relative order W/W_0 are neglected. For narrow cores ($R/Z \ll 1$) we require $ZW_0/\nu \gg 1$, and thus the similarity regimes will occur

† It may be recalled that vortex tubes cannot actually terminate at a boundary but connect into a boundary flow; thus, although the concentrated core terminates at the boundary, its constituent vortex tubelets curve into the terminating boundary layer.

at distances $Z \gg \nu/W_0$ from the virtual source. In many cases the core flow will be turbulent, in spite of the stabilizing effect of rotation, as a result of the axial flow. In strongly rotating laminar cores the centrifugal pressure field plays a significant role in the core dynamics (and we shall distinguish cores as strong or weak according to whether the axial and azimuthal velocity fields are or are not pressure coupled), and

$$P \sim \rho W_0 W, \quad V^2 \sim W_0 W; \tag{24}$$

in weakly rotating cores the pressure variations may be neglected with consequent decoupling of the axial and azimuthal velocities, and

$$V^2 < W_0 W(R/Z)^2, \quad P \sim \rho W_0 W(R/Z)^2. \tag{25}$$

The integral relations for core flows in a parallel stream are

$$\frac{d}{dz}\int_0^\infty \rho(W_0-w)\,r\,dr = (\rho r u)_\infty, \tag{26}$$

$$\frac{d}{dz}\int_0^\infty \rho r^2 v w\,dr = -(\rho r^2 u v)_\infty + \left(\rho\nu r^3 \frac{\partial}{\partial r}\frac{v}{r}\right)_\infty, \tag{27}$$

$$\frac{d}{dz}\int_0^\infty \{\rho w(W_0-w)-p\}\,r\,dr = -\{\rho r u(W_0-w)\}_\infty + \left\{\rho\nu r\frac{\partial}{\partial r}(W_0-w)\right\}_\infty, \tag{28}$$

where viscous terms have been omitted from the integrands since their fractional contributions are $O(R/Z)^2$; equation (11) is unchanged. The circulation K_∞ is given by

$$K_\infty = (2\pi r v)_\infty = \int_A \operatorname{curl}\mathbf{v}.\mathbf{n}\,dA,$$

where dA is an element of area of a plane section A of the core. The similarity structure may now be determined for swirling wakes and jets in a following stream and also for the corresponding vortex flows.

For swirling cores in a uniform stream ru remains finite, whilst rv and $(W_0-w)\to 0$ as $r\to\infty$: hence (27) and (28) reduce to

$$2\pi\int_0^\infty \rho r^2 v w\,dr = G,$$

$$2\pi\int_0^\infty \{\rho w(W_0-w)-p\}\,r\,dr = F,$$

or, with additional relative error of order W/W_0,

$$2\pi\int_0^\infty \{\rho W_0(W_0-w)-p\}\,r\,dr = F,$$

representing the conservation of angular momentum flux and flow force, respectively, along the core. Hence $\rho R^3 V W \sim G$ and $\rho R^2 W_0 W \sim F$, and

$$\frac{V}{W_0} \sim \frac{G}{RF}. \tag{29}$$

In strongly swirling cores (with $P \sim \rho W_0 W$ and $K_\infty = 0$),

$$\left.\begin{array}{l} R \sim \left(\dfrac{\nu Z}{W_0}\right)^{\frac{1}{2}}, \quad U \sim \dfrac{F}{\rho\nu^2}\left(\dfrac{ZW_0}{\nu}\right)^{-\frac{3}{2}} W_0, \quad V \sim \left(\dfrac{F}{\rho\nu^2}\right)^{\frac{1}{2}}\left(\dfrac{ZW_0}{\nu}\right)^{-\frac{1}{2}} W_0, \\ W \sim \dfrac{F}{\rho\nu^2}\left(\dfrac{ZW_0}{\nu}\right)^{-1} W_0, \quad P \sim \rho W_0^2 \dfrac{F}{\rho\nu^2}\left(\dfrac{ZW_0}{\nu}\right)^{-1}. \end{array}\right\} \tag{30}$$

In this case the swirl number

$$\frac{G}{RF} \sim \left(\frac{F}{\rho\nu^2}\right)^{\frac{1}{2}}\left(\frac{ZW_0}{\nu}\right)^{-\frac{1}{2}}$$

decreases downstream as $Z^{-\frac{1}{2}}$, in contrast with the Z^{-1} decay in a still environment; however, the role of the centrifugal pressure does not reduce correspondingly, and initial strong swirl remains strong (in the sense that axial and azimuthal velocities are pressure coupled) at all distances downstream. We note that

$$\frac{P}{\rho W_0^2} \sim \frac{W}{W_0} \sim \frac{V^2}{W_0^2} \sim \left(\frac{G}{RF}\right)^2 \sim \frac{F}{\rho\nu^2}\left(\frac{ZW_0}{\nu}\right)^{-1}, \tag{31}$$

and it follows that in the region under discussion the swirl number will not exceed about one third. After rearrangement of (31)

$$F \sim \rho(W_0 G/\rho)^{\frac{2}{3}}, \tag{32}$$

which may be interpreted as a scale for the drag force of strongly swirling wakes or jets in an ambient stream.

In weakly swirling cores the flow force contribution from the centrifugal pressure is negligible and the core acts as a channel for the forced convection of angular momentum. The scales for R, U and W remain as in (30), but

$$P \sim \rho W_0^2 \frac{F}{\rho\nu^2}\left(\frac{ZW_0}{\nu}\right)^{-2}, \quad V \sim \frac{G}{RF} W_0. \tag{33}$$

In both cases the amplitude of the azimuthal velocity decays downstream with lateral viscous growth of the core, but the dynamical role of the centrifugal pressure remains unchanged in each similarity regime, and strong swirl (according to our definitions) does not necessarily decay to weak swirl. It may be worth noting that although similarity arguments are extremely useful in establishing guidelines for certain problems, the flows observed experimentally are often intermediate in character between two similarity solutions; Batchelor (1964) has given such an intermediate solution for the vortex wake.

Freely moving bodies in steady motion cannot generate swirling wakes (jets, or self-propelled wakes) as this would result in a torque on the body and in its continued angular acceleration, though they can produce pairs of wakes with equal and opposite fluxes of angular momentum; however, swirling core flows can be produced by bodies or sources which are rigidly supported in a stream. Steiger & Bloom (1962) have described an integral method of treatment and given some approximate solutions for strongly swirling laminar wakes.

Vortex wakes in otherwise irrotational streams occur commonly in matched pairs with finite net energy and angular momentum, and constitute an essential ingredient of aerodynamic lift on moving bodies. In the early stages of core formation the circulation K_∞ increases with increasing distance downstream as each separated vortex sheet winds up to form a core; there is a progressive decrease in core pressure with consequent forward acceleration of the core fluid to velocities perhaps considerably in excess of ambient (Hall 1966). When the process of core formation has been completed, the circulation remains constant with further increase of downstream distance, and in this region of slow lateral diffusion the core flow suffers deceleration on account of the decreasing centrifugal pressure deficit. Perhaps further downstream again, vorticity starts to diffuse across the plane of symmetry separating the two vortices of the pair, with the result that thereafter neither the core circulation K_∞ nor the flux of angular momentum G are conserved for a single vortex core; and far downstream the two vortices very largely overlap and must be treated singly as a vortex line doublet. Thus the axial flow in a vortex wake may be in either sense, and there is little point in distinguishing between sink and source flows.

Relations (23) and (24) or (25) apply also to vortex wakes, but as $(rv)_\infty$ is nonzero the flux of angular momentum is not conserved. Thus a vortex wake is characterized by its flow force F and net circulation K_∞, both of which are conserved provided that the companion vortex is sufficiently far off to neglect diffusion of vorticity across the mid-plane: in this case the conservation relations imply the scaling,

$$R^2 W_0 W \sim F/\rho, \quad RV \sim K_\infty. \tag{34}$$

In strong vortex wakes $\qquad P \sim \rho W_0 W,$

and

$$\left.\begin{aligned} R &\sim \left(\frac{\nu Z}{W_0}\right)^{\frac{1}{2}}, \quad U \sim \left(\frac{K_\infty}{\nu}\right)^2 \left(\frac{ZW_0}{\nu}\right)^{-\frac{3}{2}} W_0, \quad V \sim \frac{K_\infty}{\nu}\left(\frac{ZW_0}{\nu}\right)^{-\frac{1}{2}} W_0, \\ W &\sim \left(\frac{K_\infty}{\nu}\right)^2 \left(\frac{ZW_0}{\nu}\right)^{-1} W_0, \quad P \sim \rho W_0^2 \left(\frac{K_\infty}{\nu}\right)^2 \left(\frac{ZW_0}{\nu}\right)^{-1}; \end{aligned}\right\} \tag{35}$$

and

$$\frac{F}{\rho \nu^2} \sim \left(\frac{K_\infty}{\nu}\right)^2, \tag{36}$$

which may be interpreted as a scale for the flow force of a vortex wake. The direct dependence of flow force on circulation arises from the strong coupling between the flow components; it may be noted also that K_∞/ν and $(F/\rho\nu^2)^{\frac{1}{2}}$ are effectively alternative forms of the total Reynolds number. This similarity regime applies only where $ZW_0/\nu \gg 1$ and $ZW_0/\nu \gg (K_\infty/\nu)^2$, and once achieved it extends indefinitely downstream. In weak vortex wakes the azimuthal velocity scales with the circulation, but decouples from the remaining components of the flow, which scale with the flow force:

$$R \sim \left(\frac{\nu Z}{W_0}\right)^{\frac{1}{2}}, \quad U \sim \frac{F}{\rho\nu^2}\left(\frac{ZW_0}{\nu}\right)^{-\frac{3}{2}} W_0, \quad V \sim \frac{K_\infty}{\nu}\left(\frac{ZW_0}{\nu}\right)^{-\frac{1}{2}} W_0,$$

$$W \sim \frac{F}{\rho\nu^2}\left(\frac{ZW_0}{\nu}\right)^{-1} W_0, \quad P \sim \rho W_0^2 \frac{F}{\rho\nu^2}\left(\frac{ZW_0}{\nu}\right)^{-2}.$$

The wake will again be narrow provided that $ZW_0/\nu \gg 1$ and the disturbance velocity will be relatively small (the wake approximation) if $F/\rho\nu^2 \ll ZW_0/\nu$; and the vortex will be weak if the centrifugal contribution to the radial pressure gradient is smaller than that due to convection, or

$$\frac{K_\infty}{\nu} < \left(\frac{F}{\rho\nu^2}\right)^{\frac{1}{2}} \left(\frac{ZW_0}{\nu}\right)^{-\frac{1}{2}}.$$

A solution in closed form for weak trailing vortices has been given by Newman (1959).

The foregoing discussion of laminar swirling and vortex core flows serves to illuminate Reynolds's (1962) treatment of swirling turbulent wakes. Reynolds neglects the pressure field entirely, thereby implicitly restricting his argument to weakly swirling flows and ensuring that the axial and azimuthal fields decouple: thus his treatment cannot apply to 'swirl-dominated flows' at all. He imposes conservation of linear momentum flux (rather than flow force), whereas the flux of simple linear momentum is conserved only in the absence of angular momentum flux, which itself is conserved only in the absence of circulation. We note also that a self-propelled body in a homogeneous environment cannot generate a single swirling wake without suffering progressive angular acceleration about its line of motion, though a non-lifting body can generate a pair of wakes having opposed rotation. A single swirling wake can, however, be generated by a ship or other floating body propelled along a density discontinuity surface in a laterally inclined state.

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